Available online at www.sciencedirect.com
ScienceDirect
Journal of Computational and Applied Mathematics 212 (2008) 245–259
JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS
www.elsevier.com/locate/cam

Structured pseudospectra for nonlinear eigenvalue problems

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Received 30 May 2006; received in revised form 5 September 2006

Abstract

In this paper we introduce structured pseudospectra for nonlinear eigenvalue problems and derive computable formulae. The results are applied to the sensitivity analysis of the eigenvalues of a second-order system arising from structural dynamics and of a time-delay system arising from laser physics. In the former case, a comparison is made with the results obtained in the framework of random eigenvalue problems.

MSC: 47J10; 15A18; 34L16

Keywords: Nonlinear eigenvalue problem; Structured pseudospectra; Robustness; Structured singular values

1. Introduction

Pseudospectra have recently found application in analysing the sensitivity of eigenvalues of a system [8,17]. Principally, pseudospectra are sets in the complex plane to which the eigenvalues of a system can be shifted, under a random perturbation of a given size. In this way, one can classify the degree of sensitivity of the system's eigenvalues. Moreover, for robust stability, the pseudospectra identify the minimum size of a random perturbation required to shift an eigenvalue such that stability is lost. In this case, one may directly compare the size of the perturbation with the stability radius of the system [4,9,12,14].

Mathematically, in the simplest setting, given a matrix $A \in \mathbb{C}^{n\times n}$ one can investigate the sensitivity of its eigenvalues under additive perturbations by considering the pseudospectra (or spectral value sets)

$$\begin{aligned}\Lambda_\varepsilon(A) &= \{\lambda \in \mathbb{C} : \lambda \in \Sigma(A+P) \text{ for some } P \in \mathbb{C}^{n\times n} \text{ with } \|P\| < \varepsilon\}\\ &= \{\lambda \in \mathbb{C} : \|(A-\lambda I_n)^{-1}\| > 1/\varepsilon\},\end{aligned}$$

where I_n denotes the $n \times n$-identity matrix, and $\Sigma(\cdot)$ denotes the spectrum [16,17].

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doi:10.1016/j.cam.2006.12.007

In a number of problems the matrix A has a certain structure, for example, a block structure, which should be respected in the sensitivity analysis. For this, perturbations of the form $A + DPE$ are considered in Ref. [7], where the fixed matrices D and E describe the perturbation structure and P is a complex perturbation matrix. This approach has been further developed in Ref. [19] for perturbations of the form $A + \sum D_i P_i E_i$, which, in particular, allow one to deal with element-wise perturbations.

On the other hand, specific classes of systems, like higher order systems or systems with time delays, lead to the study of the zeros of *nonlinear eigenvalue problems* of the form

$$F(\lambda) := \sum_{i=1}^{m} A_i p_i(\lambda), \tag{1}$$

where $p_i, i = 1, \ldots, m$, are entire functions. For example, the characteristic matrix of the second-order system $A_3\ddot{x}(t) + A_2\dot{x}(t) + A_1x(t) = 0$ is given by $A_3\lambda^2 + A_2\lambda + A_1$ and the characteristic matrix of the time-delay system $\dot{x}(t) = A_1x(t) + A_2x(t-\tau)$ by $\lambda I - A_1 - A_2\mathrm{e}^{-\lambda\tau}$. Although such systems can usually be rewritten in a first-order form, it is advantageous to exploit the structure of the governing equation. Pseudospectra for polynomial matrices were introduced in Ref. [14]. A discussion of pseudospectra for general nonlinear eigenvalue problems of form (1) has been presented in Ref. [9]. The latter reference deals with the distribution of zeros of $\sum_{i=1}^{m}(A_i + \delta A_i)p_i(\lambda)$, where the δA_i are complex, unstructured perturbation matrices, and a suitable joint norm for these perturbation matrices is used in the definitions of pseudospectra.

The goal of this study is to combine the above two approaches for exploiting a system's structure. In light of this, we define pseudospectra for the nonlinear eigenvalue problem (1) and derive computable formulae, where, in addition to exploiting the form of the nonlinear eigenvalue problem, a particular structure can be imposed on the perturbations of the individual coefficient matrices A_i.

The motivation stems from the fact that in a lot of applications the coefficient matrices have a certain structure that should naturally be respected in a sensitivity analysis, since unstructured perturbations may lead to irrelevant or non-physical effects. One example is discussed in Ref. [5], where the emergence of unbounded pseudospectra of a delay system in certain directions is explained by non-physical perturbations that destroy an intrinsic property, namely the singular nature, of one of the coefficient matrices. Other motivating examples from areas of application will be discussed in Section 3.

The main mathematical tool to arrive at computable formulae is a reformulation of the sensitivity problem in terms of *structured singular values* (ssv); see the Appendix, or Refs. [8,10] for more details. For a broad class of perturbation structures a general computable expression for the corresponding pseudospectra is derived. This involves the calculation of appropriately defined ssv. It is outlined in which cases such ssv can be computed exactly or how bounds can be derived otherwise. Next, it is illustrated how relaxing the perturbation structure may lead to exact and more efficient computable formulae. This allows one to weigh the advantages of imposing structure versus computational complexity, which is relevant from an application point of view.

The structure of the paper is as follows. In Section 2 structured pseudospectra for nonlinear eigenvalue problems are defined and computable formulae are derived. Section 3 describes practical applications from structural mechanics and laser physics. Section 4 contains the conclusions. The Appendix is devoted to some background material on the ssv.

2. Structured pseudospectra for nonlinear eigenvalue problems

2.1. Motivation and definition

Following the work of Ref. [9], we are interested in general nonlinear eigenvalue problems of form (1), where $A_i \in \mathbb{C}^{n\times n}$ and $p_i : \mathbb{C} \to \mathbb{C}$ is an entire function, for all $i = 1, \ldots, m$. In what follows, we call $F(\lambda)$ the *characteristic matrix* and refer to the zeros of $\det(F(\lambda)) = 0$ as the *eigenvalues* of F. We denote the *spectrum* of F as

$$\Lambda := \{\lambda \in \mathbb{C} : \det(F(\lambda)) = 0\}. \tag{2}$$

A definition for the ε-*pseudospectrum* of the nonlinear eigenvalue problem (1) is given in Ref. [9] as

$$\Lambda_\varepsilon(F) := \left\{ \lambda \in \mathbb{C} : \det\left(\sum_{i=1}^{m} (A_i + \delta A_i)\, p_i(\lambda) \right) = 0, \text{ for some } \delta A_i \in \mathbb{C}^{n\times n} \right.$$
$$\left. \text{with } w_i \|\delta A_i\|_2 < \varepsilon, \ 1 \leqslant i \leqslant m \right\}, \tag{3}$$

where $w_i > 0$ are weights and $\|\cdot\|_2$ denotes the 2-norm of a matrix. Denoting the largest singular value of a matrix by σ_1 we have $\|\cdot\|_2 = \sigma_1(\cdot)$. We observe that the perturbations δA_i considered in (3) lead to an additive uncertainty on the characteristic matrix (1) given by

$$\delta F(\lambda) := \sum_{i=1}^{m} \delta A_i \ p_i(\lambda). \tag{4}$$

Although the structure of expression (1) is explicitly taken into account in definition (3), the perturbations δA_i applied to the different matrices A_i are unstructured. In other words, the element-wise structure of A_i is not taken into account when using the corresponding perturbation δA_i.

The goal of this section is to present a framework for the definition and computation of pseudospectra, in which various types of structure on the perturbation matrices can also be imposed. For this we assume a more general additive uncertainty on (1) than what (4) allows. This uncertainty takes the form

$$\delta F(\lambda) := \sum_{j=1}^{f} D_j(\lambda) \Delta_j E_j(\lambda) + \sum_{j=1}^{s} d_j G_j(\lambda) H_j(\lambda). \tag{5}$$

In this expression $\Delta_j \in \mathbb{C}^{k_j \times l_j}$ and $d_j \in \mathbb{C}$ denote the underlying unstructured perturbations, and $D_j \in \mathbb{C}^{n\times k_j}$, $E_j \in \mathbb{C}^{l_j \times n}$, $G_j \in \mathbb{C}^{n\times m_j}$ and $H_j \in \mathbb{C}^{m_j \times n}$ are appropriate shape matrices, whose elements are entire functions. We further assume that $m_j \geqslant 2$, G_j has full column rank and H_j has full row rank, for all $j = 1, \ldots, s$. Note that, for the first assumption, terms in the second summation of (5) with $m_j = 1$ can be incorporated into the first summation. Furthermore, if the second assumptions were not fulfilled, one could always decompose $G_j H_j$ as, say, $\tilde{G}_j \tilde{H}_j$ with $\tilde{G}_j$ having full column rank (or $\tilde{H}_j$ having full row rank). This also has the consequence that we always deal with an uncertainty of the smallest dimension m_j. The same assumptions cannot be placed on D_j or E_j as a rank reduction would require redefining Δ_j.

The structured ε-pseudospectrum $\Lambda_\varepsilon^{\mathrm{str}}(F)$ of F with respect to the uncertainty (5) can then be defined as follows:

Definition 1.

$$\Lambda_\varepsilon^{\mathrm{str}}(F) := \{\lambda \in \mathbb{C} : \det(F(\lambda) + \delta F(\lambda)) = 0, \text{ for some } \delta F \text{ as in (5)}$$
$$\text{with } \|\Delta_j\|_2 < \varepsilon, \ 1 \leqslant j \leqslant f \text{ and } |d_j| < \varepsilon, \ 1 \leqslant j \leqslant s\}. \tag{6}$$

Note that the weights w_i, given in (3), are absorbed into the elements of the shape matrices D_j, E_j, G_j and H_j, given in (5).

Before presenting a method for computing $\Lambda_\varepsilon^{\mathrm{str}}(F)$, we motivate our choice of the uncertainty structure (5) and discuss how certain problems fit into this general framework:

- In Ref. [12] the authors deal with perturbations of matrix polynomials $\sum_{i=1}^{m} A_i\, p_i(\lambda)$ of the form

$$\sum_{i=1}^{m} \left(A_i + \sum_{j=1}^{f} D_j \Delta_j E_{ij} \right) p_i(\lambda) \tag{7}$$

  in the context of stability radii for polynomial matrices. The shape matrices D_j and E_{ij} in (7) can be used to perturb only a submatrix of A_i, to assign weights to perturbations of rows, columns or elements of each A_i, and to weight

the perturbations applied to the matrices $A_1, \ldots, A_m$ with respect to each other. The uncertainty in (7) leads to the additive perturbation

$$\delta F(\lambda) = \sum_{j=1}^{f} \left[D_j \Delta_j \left(\sum_{i=1}^{m} E_{ij} p_i(\lambda) \right) \right].$$

- In many applications the characteristic matrix of an uncertain system is given by $\sum_{i=1}^{m} \tilde{A}_i p_i(\lambda)$, where the matrices $\tilde{A}_i$ depend linearly on a number of uncertain scalar parameters, say

$$\tilde{A}_i = A_i + \sum_j \theta_j P_{ij}$$

with $\theta_j \in \mathbb{C}$ describing the uncertainties on these parameters. Assuming that one wishes to investigate the possible positions of the eigenvalues when $|\theta_j| \leqslant \varepsilon, \forall j$. It follows that we are in the framework of (1), (5) and (6) as we can express

$$\begin{aligned} \sum_{i=1}^{m} \tilde{A}_i p_i(\lambda) &= F(\lambda) + \sum_j \theta_j \left(\sum_{i=1, P_{ij} \neq 0}^{m} U_{ij} V_{ij}^* p_i(\lambda) \right) \\ &= F(\lambda) + \sum_j \theta_j [\cdots U_{ij} \cdots][\cdots V_{ij} \bar{p}_i(\lambda) \cdots]^*, \end{aligned} \tag{8}$$

where each U_{ij} has full column rank and U_{ij} and V_{ij} can be computed, for instance, from a singular value decomposition of P_{ij}. Notice that (8) leads to $s > 0$ in the general expression (5) if and only if one of the matrices P_{ij} has rank larger than one, or if one of the parameters explicitly appears in different matrices $\tilde{A}_i$.

- Structured perturbations of the form (5) can sometimes also be used for systems with a nonlinear dependence on the parameters. As an illustration, the uncertain system

$$\dot{x}(t) = (A + \delta A)x(t) + (B + \delta B)(C + \delta C)x(t - \tau)$$

can be rewritten in a descriptor form as

$$\begin{aligned} \dot{x}(t) &= (A + \delta A)x(t) + (B + \delta B)y(t), \\ 0 &= (C + \delta C)x(t - \tau) - y(t). \end{aligned}$$

This has a nominal characteristic matrix

$$F(\lambda) = \begin{bmatrix} \lambda I - A & -B \\ C\mathrm{e}^{-\lambda\tau} & -I \end{bmatrix}$$

to which we may apply the structured perturbations

$$\delta F(\lambda) = -\begin{bmatrix} I \\ 0 \end{bmatrix} \delta A [I \ 0] - \begin{bmatrix} I \\ 0 \end{bmatrix} \delta B [0 \ I] - \begin{bmatrix} 0 \\ I \end{bmatrix} \delta C [\mathrm{e}^{-\lambda\tau} I \ 0].$$

2.2. Formulation as an ssv problem

We now derive a computable formula for $\Lambda_\varepsilon^{\mathrm{str}}(F)$. An important observation is that (6) can be reformulated in terms of ssv. A brief introduction to this concept can be found in the Appendix; now we will only recall the definition from Ref. [10]:

Let $\mathbf{\Delta}$ be a closed subset of $\mathbb{C}^{M \times N}$, the so-called uncertainty set. Then for a matrix $G \in \mathbb{C}^{N \times M}$ its ssv $\mu_{\mathbf{\Delta}}(G)$ with respect to $\mathbf{\Delta}$ is defined as

$$\mu_{\mathbf{\Delta}}(G) := \begin{cases} 0 & \text{if } \det(I + G\Delta) \neq 0 \ \forall \Delta \in \mathbf{\Delta}, \\ (\min\{\sigma_1(\Delta) : \Delta \in \mathbf{\Delta} \text{ and } \det(I + G\Delta) = 0\})^{-1} & \text{otherwise.} \end{cases}$$

We make use of this concept in the following proposition.

Theorem 1. *Consider the characteristic matrix* (1) *with additive uncertainty* (5). *We define the uncertainty set* $\mathbf{\Delta}$ *as*

$$\mathbf{\Delta} := \{\mathrm{diag}(\Delta_1, \ldots, \Delta_f, d_1 I_{m_1}, \ldots, d_s I_{m_s}) : \Delta_i \in \mathbb{C}^{k_i \times l_i},\ d_j \in \mathbb{C},\ 1 \leqslant i \leqslant f,\ 1 \leqslant j \leqslant s\}, \tag{9}$$

where diag(·) *represents a block-diagonal matrix. Let*

$$M(\lambda) := [E_1(\lambda)^{\mathrm{T}} \ldots E_f(\lambda)^{\mathrm{T}} H_1(\lambda)^{\mathrm{T}} \ldots H_s(\lambda)]^{\mathrm{T}},$$
$$N(\lambda) := [D_1(\lambda) \cdots D_f(\lambda) G_1(\lambda) \cdots G_s(\lambda)]$$

and

$$T(\lambda) := M(\lambda) F(\lambda)^{-1} N(\lambda).$$

Then

$$\Lambda_\varepsilon^{\mathrm{str}}(F) = \Lambda \cup \left\{ \lambda \in \mathbb{C} : \mu_{\mathbf{\Delta}}(T(\lambda)) > \frac{1}{\varepsilon} \right\}, \tag{10}$$

where $\mu_{\mathbf{\Delta}}(\cdot)$ *is the ssv with respect to the uncertainty set* $\mathbf{\Delta}$ *defined in* (9).

Proof. If $\det(F(\lambda)) \neq 0$ we have the following equivalence:

$$\begin{aligned}
&\det(F(\lambda) + \delta F(\lambda)) = 0 \\
&\Updownarrow \\
&\det\left(I + F(\lambda)^{-1} [D_1(\lambda) \cdots D_f(\lambda) G_1(\lambda) \cdots G_s(\lambda)] \underline{\Delta} \begin{bmatrix} E_1(\lambda) \\ \vdots \\ E_f(\lambda) \\ H_1(\lambda) \\ \vdots \\ H_s(\lambda) \end{bmatrix} \right) = 0 \\
&\Updownarrow \\
&\det(I + T(\lambda)\underline{\Delta}) = 0
\end{aligned} \tag{11}$$

for some matrix $\underline{\Delta} = \mathrm{diag}(\Delta_1, \ldots, \Delta_f, d_1 I, \ldots, d_s I) \in \mathbf{\Delta}$.

Furthermore,

$$\|\underline{\Delta}\|_2 < \varepsilon \quad \Leftrightarrow \quad \|\Delta_j\|_2 < \varepsilon, \quad 1 \leqslant j \leqslant f \quad \text{and} \quad |d_j| < \varepsilon, \quad 1 \leqslant j \leqslant s. \tag{12}$$

Considering (11) and (12) with the definition of $\Lambda_\varepsilon^{\mathrm{str}}$, it follows that if $\lambda \in \Lambda_\varepsilon^{\mathrm{str}}$, then either $\lambda \in \Lambda$ or the following holds:

$$\exists \underline{\Delta} \in \mathbf{\Delta} \text{ with } \|\underline{\Delta}\|_2 < \varepsilon \text{ such that } \det(I + T(\lambda)\underline{\Delta}) = 0.$$

Hence,

$$\min\{\|\Delta\|_2 : \Delta \in \mathbf{\Delta} \text{ and } \det(I + T(\lambda)\Delta) = 0\} < \varepsilon$$

which implies $\mu_{\mathbf{\Delta}}(T(\lambda)) > \varepsilon^{-1}$. □

Subsequently, from (10) the boundaries of structured ε-pseudospectra can be defined as level sets of the function

$$\mu_{\mathbf{\Delta}}(T(\lambda)), \quad \lambda \in \mathbb{C}. \tag{13}$$

In general, the computation of the ssv of a matrix with respect to the uncertainty set (9) is a difficult problem. Indeed, it is known to belong to the class of NP hard problems [15], which makes it inefficient for large problems. In many cases,

however, the efficiency of the ssv computation can be increased by exploiting the structure of $T(\lambda)$. Some results in this sense will be discussed in Section 2.3.

Furthermore, for the examples presented in Section 3, we found that numerical algorithms that approximate the ssv performed well. There, we compute lower and upper bounds on the ssv by solving eigenvalue optimization problems. Such bounds are sharp in many cases. If the additional restriction $f + 2s \leqslant 3$ holds for the uncertainty set (9) and the full blocks are square, then an exact computation of $\mu_{\mathbf{\Delta}}(\cdot)$ is always possible by solving a convex optimization problem; see the Appendix, Refs. [10,20] and the references therein.

Hence, we propose the use of the ssv for computing structured pseudospectra. Note in particular that the computation of upper bounds for ssv (13) gives lower bounds for the ε-values, for which a point lies in the structured ε-pseudospectrum. This is in agreement with the common use of pseudospectra (or directly related values like stability radii) in a worst-case analysis. In this context the above approach can be used to give rigorous bounds for the behaviour of eigenvalues under perturbations.

2.3. Special cases

In some cases the particular structure of $T(\lambda)$ can be exploited for an efficient computation of ssv (13). This is illustrated by the following result, which generalizes one of the assertions of Theorem 1 of Ref. [9] and is also related to Proposition 3.4 of Ref. [12]:

Theorem 2. *We consider the characteristic matrix* (1) *with uncertainty* (5). *Furthermore, we assume that* $s = 0$, *and that there exist matrices D and E and functions* $q_j : \mathbb{C} \to \mathbb{C}$ *such that*

$$D_j(\lambda) = D(\lambda),$$
$$E_j(\lambda) = E(\lambda) q_j(\lambda), \quad 1 \leqslant j \leqslant f.$$

By defining $T(\lambda)$ *and* $\mathbf{\Delta}$ *as in Theorem* 1, *the following holds*:

$$\mu_{\mathbf{\Delta}}(T(\lambda)) = \|E(\lambda)F^{-1}(\lambda)D(\lambda)\|_2 \left(\sum_{j=1}^{f} |q_j(\lambda)| \right), \quad \lambda \notin \Lambda. \tag{14}$$

Proof. Since $\det(F(\lambda)) \neq 0$, the statement $\det(F(\lambda) + \delta F(\lambda)) = 0$ is equivalent with

$$\det \left(I + E(\lambda)F(\lambda)^{-1}D(\lambda) \sum_{j=1}^{f} \Delta_j q_j(\lambda) \right) = 0. \tag{15}$$

Note that all Δ_i have the same dimension, say k times l. If (15) is satisfied, then

$$1 \leqslant \|E(\lambda)F(\lambda)^{-1}D(\lambda)\|_2 \left\| \sum_{j=1}^{f} q_j(\lambda)\Delta_j \right\|_2$$
$$\leqslant \|E(\lambda)F(\lambda)^{-1}D(\lambda)\|_2 \left(\sum_{j=1}^{f} |q_j(\lambda)| \right) \max_{1 \leqslant j \leqslant f} \|\Delta_j\|_2,$$

hence,

$$\max_{1 \leqslant j \leqslant f} \|\Delta_j\|_2 \geqslant \left(\|E(\lambda)F(\lambda)^{-1}D(\lambda)\|_2 \left(\sum_{j=1}^{f} |q_j(\lambda)| \right) \right)^{-1}. \tag{16}$$

Furthermore, there exists a complex matrix U, see, for example, [10] or Appendix A, property (28), such that

$$\det(I - E(\lambda)F(\lambda)^{-1}D(\lambda)\ U) = 0, \quad \|U\|_2 = (\|E(\lambda)F(\lambda)^{-1}D(\lambda)\|_2)^{-1}.$$

Now let

$$\Delta_j^{\mathrm{c}} = \frac{-\bar{q}_j(\lambda)}{|q_j(\lambda)|\sum_{j=1}^{f}|q_j(\lambda)|}\,U, \quad j = 1,\ldots,f.$$

It is easy to check that with $(\Delta_1,\ldots,\Delta_f) = (\Delta_1^{\mathrm{c}},\ldots,\Delta_f^{\mathrm{c}})$ (15) is satisfied, and, in addition, the equality is reached in (16). It follows that

$$\min\left\{\max_{1\leqslant j\leqslant f}\|\Delta_j\|_2 :\ \Delta_j\in\mathbb{C}^{k\times l},\ 1\leqslant j\leqslant f \text{ and (15) is satisfied}\right\}$$
$$= \left(\|E(\lambda)F(\lambda)^{-1}D(\lambda)\|_2\ \left(\sum_{j=1}^{f}|q_j(\lambda)|\right)\right)^{-1}.$$

Finally, the left-hand side can be expressed as

$$\min\{\|\Delta\|_2 : \Delta\in\mathbf{\Delta} \text{ and } \det(I - T(\lambda)\Delta) = 0\} = (\mu_{\mathbf{\Delta}}(T(\lambda)))^{-1}$$

and we arrive at the statement of the proposition. □

We note that, in addition to the availability of a directly computable formula, expressed in terms of singular values instead of ssv, the dimensions of $E(\lambda)F^{-1}(\lambda)D(\lambda)$ are *f times smaller* than the dimensions of $T(\lambda)$. Thus, reducing computational costs.

It is useful to discuss special types of problems for which the combination of Theorems 1 and 2 ensures an efficient computation of pseudospectra. Hence, we conclude this section with an application of Theorem 2 to various types of uncertainty structures proposed in the pseudospectra literature:

- Unstructured pseudospectra, in the sense of Ref. [9].
  For $F(\lambda) = \sum_{i=1}^{m}(A_i + \delta A_i)p_i(\lambda)$ we have

  $$\delta F(\lambda) = \sum_{i=1}^{m}\underbrace{I}_{D_i(\lambda)}\underbrace{\delta A_i}_{\Delta_i}\underbrace{p_i(\lambda)I}_{E_i(\lambda)},$$

  and hence

  $$\mu_{\mathbf{\Delta}}(T(\lambda)) = \|F(\lambda)^{-1}\|_2\sum_{i=1}^{m}|p_i(\lambda)|. \tag{17}$$

- Pseudospectra in the sense of Ref. [14].
  For $F(\lambda) = \sum_{i=1}^{m}(A_i + D\delta A_i E_i)p_i(\lambda)$ we have

  $$\delta F(\lambda) = \underbrace{D}_{D(\lambda)}\underbrace{[\delta A_1\ldots\delta A_m]}_{\Delta}\underbrace{[E_1^{\mathrm{T}}p_1(\lambda)\cdots E_m^{\mathrm{T}}p_m(\lambda)]^{\mathrm{T}}}_{E(\lambda)}$$

  and by Theorem 2

  $$\mu_{\mathbf{\Delta}}(T(\lambda)) = \left\|\begin{bmatrix}p_1(\lambda)E_1\\ \vdots\\ p_m(\lambda)E_m\end{bmatrix}F(\lambda)^{-1}D\right\|_2.$$

  For the special case $E_i = E$, $i = 1,\ldots,m$, this expression can be simplified to

  $$\mu_{\mathbf{\Delta}}(T(\lambda)) = \|EF^{-1}(\lambda)D\|_2\sqrt{\sum_{i=1}^{m}|p_i(\lambda)|^2}. \tag{18}$$

Furthermore, if we consider the dual problem, $F(\lambda) = \sum_{i=1}^{m} (A_i + D_i \delta A_i E) p_i(\lambda)$, instead, we have

$$\delta F(\lambda) = \underbrace{[D_1 p_1(\lambda) \cdots D_m p_m(\lambda)]}_{D(\lambda)} \underbrace{[\delta A_1^{\mathrm{T}} \cdots \delta A_m^{\mathrm{T}}]^{\mathrm{T}}}_{\Delta} \underbrace{E}_{E(\lambda)}$$

and

$$\mu_{\Delta}(T(\lambda)) = \|EF(\lambda)^{-1}[p_1(\lambda)D_1 \cdots p_m(\lambda)D_m]\|_2,$$

which can (of course) for the special case $D_i = D,\ i = 1, \ldots, m$, be simplified to (18).

Notice that expression (18) does not reduce to expression (17) if $D = I$ and $E = I$, although in that case they concern the same additive perturbation on F. This is explained by the fact that both formulae are based on a different way of measuring the perturbations ($\max_{1 \leqslant i \leqslant m} \|\delta A_i\|_2$ for (17) and $\|[\delta A_1 \ldots \delta A_m]\|_2$ for (18)).

3. Applications

We now use the theory developed in Section 2 to analyse the sensitivity of eigenvalues in two physical systems. The first example, from structural dynamics, is of an undamped spring–mass system [1]. This leads to studying structured pseudospectra of a second-order system. Our second example, from laser physics, is of a semiconductor laser subject to optical feedback [18], leading to a study of structured pseudospectra of delay differential equations.

3.1. An example from structural dynamics

In Ref. [1] the effect of random perturbations on the eigenvalues of a second-order system is studied. The authors consider the three degrees-of-freedom undamped spring–mass system shown in Fig. 1.

It is described by the second-order differential equation

$$M\ddot{x}(t) + Kx(t) = 0, \tag{19}$$

where the mass matrix M and the stiffness matrix K have the following structure:

$$M = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \quad \text{and} \quad K = \begin{bmatrix} k_1 + k_4 + k_6 & -k_4 & -k_6 \\ -k_4 & k_2 + k_4 + k_5 & -k_5 \\ -k_6 & -k_5 & k_3 + k_5 + k_6 \end{bmatrix}.$$

In this example, we assume that all mass and stiffness parameters, m_i and k_i, are constant but uncertain. Specifically,

$$\begin{aligned} m_i &= \bar{m}_i(1 + \varepsilon_m x_i), \quad i = 1, \ldots, 3, \\ k_i &= \bar{k}_i(1 + \varepsilon_k x_{i+3}), \quad i = 1, \ldots, 6, \end{aligned} \tag{20}$$

where $\bar{m}_i$ and $\bar{k}_i$ are the expected values and x_i are complex random variables, whose real and imaginary parts are uncorrelated Gaussian random variables with zero mean and standard deviation one. In the numerical experiments

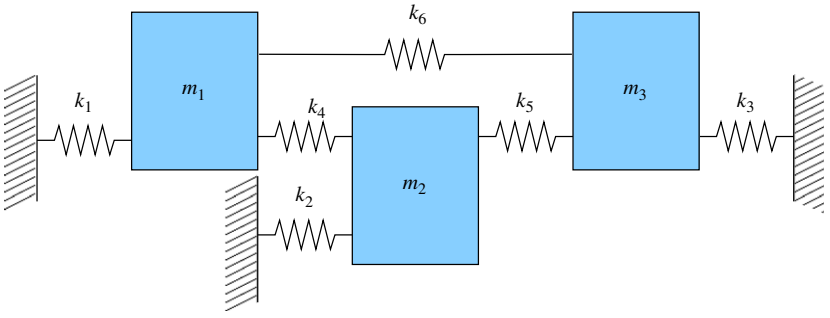

Fig. 1. A three degrees-of-freedom spring–mass system taken from Ref. [1].

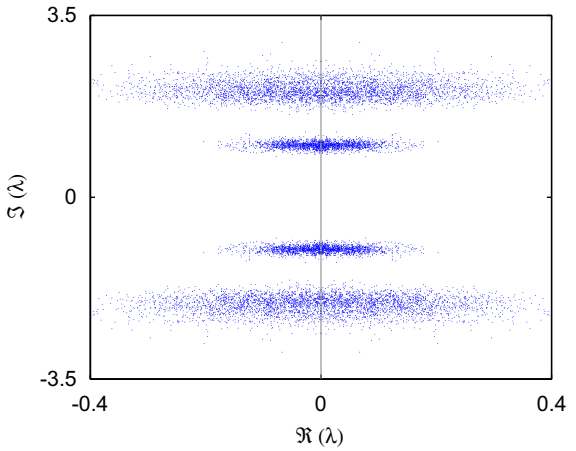

Fig. 2. Eigenvalues of 2000 simulations of the random second-order system (19).

that follow, the parameter values are taken as $\bar{m}_i = 1$, $i = 1, \ldots, 3$, $\bar{k}_i = 1$, $i = 1, \ldots 5$, $\bar{k}_6 = 1.275$ and the degree of uncertainty is described by

$$\varepsilon_m = \varepsilon_k = 0.15;$$

see the second example of Ref. [1]. The eigenvalues of (19) are the zeros of the random matrix polynomial $P(\lambda) := M\lambda^2 + K$. The characteristic matrix, obtained by taking the expectation of the parameters,

$$P_0(\lambda) := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \lambda^2 + \begin{bmatrix} 3.275 & -1 & -1.275 \\ -1 & 3 & -1 \\ -1.275 & -1 & 3.275 \end{bmatrix}, \tag{21}$$

has purely imaginary eigenvalues

$$\lambda_{\pm 1} = \pm \mathrm{i}, \quad \lambda_{\pm 2} = \pm 2\mathrm{i}, \quad \lambda_{\pm 3} = \pm 2.1331\mathrm{i}.$$

To investigate the effect of the uncertainty on the parameters given by (20) we first perform intensive random simulations. The eigenvalues of 2000 simulations are shown in Fig. 2. The eigenvalues $\lambda_{\pm 2}$ and $\lambda_{\pm 3}$ appear to be most sensitive to perturbation. Furthermore, a clear separation between the perturbations of $\lambda_{\pm 2}$ and $\lambda_{\pm 3}$ cannot be observed.

We now perform a rigorous sensitivity analysis using structured pseudospectra. Starting from the characteristic matrix (21), we express all uncertainty as an additive perturbation of form (5) as follows:

$$\begin{aligned} \delta P(\lambda) = {} & \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{D_1(\lambda)} \delta m_1 \underbrace{[1\ 0\ 0]\lambda^2}_{E_1(\lambda)} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \delta m_2 [0\ 1\ 0]\lambda^2 + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \delta m_3 [0\ 0\ 1]\lambda^2 \\ & + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \delta k_1 [1\ 0\ 0] + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \delta k_2 [0\ 1\ 0] + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \delta k_3 [0\ 0\ 1] \\ & + \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \delta k_4 [1\ -1\ 0] + \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \delta k_5 [0\ 1\ -1] + \underbrace{\begin{bmatrix} 1.275 \\ 0 \\ -1.275 \end{bmatrix}}_{D_9(\lambda)} \delta k_6 \underbrace{[1\ 0\ -1]}_{E_9(\lambda)}. \end{aligned}$$

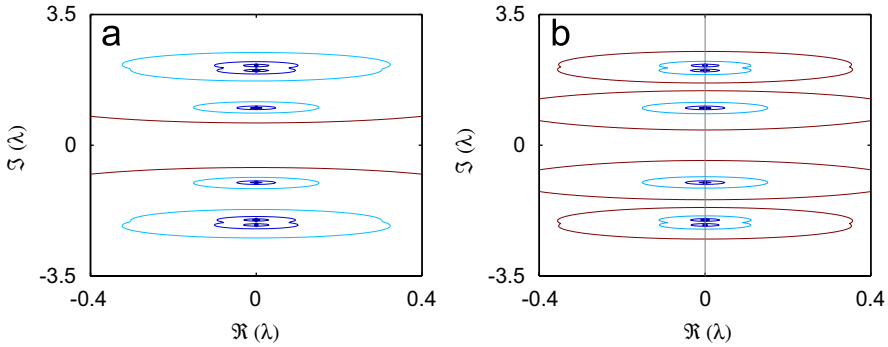

Fig. 3. Structured (a) and unstructured (b) pseudospectra of the matrix polynomial $M_0\lambda^2 + K_0$.

Observe that the weights entering the shape matrices D_i and E_i are chosen according to distribution (20). In this way pseudospectra can be computed from Theorem 1, where $\mathbf{\Delta}$ reduces to the set of complex 9×9 diagonal matrices and

$$T(\lambda) = \left[\begin{array}{c} \lambda^2 I_3 \\ \hline I_3 \\ \hline \begin{array}{ccc} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{array} \end{array}\right] P_0(\lambda)^{-1} \left[\begin{array}{c|c|c} & & \begin{array}{ccc} 1 & 0 & 1.275 \\ 1 & 1 & 0 \\ 0 & -1 & -1.275 \end{array} \\ I_3 & I_3 & \\ & & \end{array}\right]. \tag{22}$$

The computation of the structured pseudospectra is performed using the `MATLAB` routine `mussv` contained in the Robust Control Toolbox [13]. We compute $\mu_{\mathbf{\Delta}}(T(\cdot))$ on a 300×300 grid over a region of the complex plane. A contour plot then yields the boundaries of the structured pseudospectra. Note that, for the perturbation structure under consideration, only upper and lower bounds on the ssv can be computed. Along the grid, the maximum relative difference between the bounds, obtained by the function `mussv`, is of order 10^{-3}.

Fig. 3(a) shows the boundaries of structured ε-pseudospectra for $\varepsilon/0.15 = 10^{-1.5}$, 10^{-1}, $10^{-0.5}$, 1 and $10^{0.5}$. We find a good qualitative agreement with the simulations, in the sense that the eigenvalues furthest from the real axis are the most sensitive to perturbation.

To illustrate the importance of taking the structure of the perturbations into account, let us compare the results with unstructured pseudospectra of P_0 in the sense of Ref. [14]. This corresponds to definition (3). The weights of the perturbations of M and K were chosen as the 2-norm of the matrices obtained by taking the standard deviation element-wise, namely $w_M = 1/0.15$ and $w_K = 1/0.8081$. The contours of the computed pseudospectra Λ_ε are shown in Fig. 3(b), for $\varepsilon/0.15 = 10^{-1.5}$, 10^{-1}, $10^{-0.5}$ and 1. In contrast to Fig. 3(a) and the simulation results shown in Fig. 2, the eigenvalues closest to the real axis appear as the most sensitive. This indicates that unstructured pseudospectra do not adequately describe the sensitivity of eigenvalues in this problem. However, we should note that, our experiments with this system indicate that the computational cost of applying a structured perturbation is approximately 400 times that of the unstructured case.

Finally, we interpret the structured pseudospectra in a quantitative way by relating the corresponding ε-values with the uncertainty measures $\varepsilon_{m,k}$ in (20). In particular, the $\varepsilon = 0.15$ contour fits well with the simulation results shown in Fig. 2 (for $\varepsilon_{m,k}$=0.15). This correspondence is again illustrated in Fig. 4(a), where we display both the pseudospectrum contour for $\varepsilon = 0.15$ and the eigenvalues of 2000 random simulations. Thus indicating that for the system under consideration the relation $\varepsilon = \varepsilon_{m,k}$ leads to a good qualitative and quantitative agreement between both approaches. Note that ε_m and ε_k are the standard deviation of the normalized uncertain parameters, which have a Gaussian distribution, whereas ε bounds the allowable perturbations on the mean values of these parameters in the definition of the ε-pseudospectrum. This explains why some eigenvalues lie outside the pseudospectrum contour in Fig. 4(a). For comparison, Fig. 4(b) shows the boundary of the ε=0.15-pseudospectrum and the results of 2000 simulations, where it is assumed that m_i and

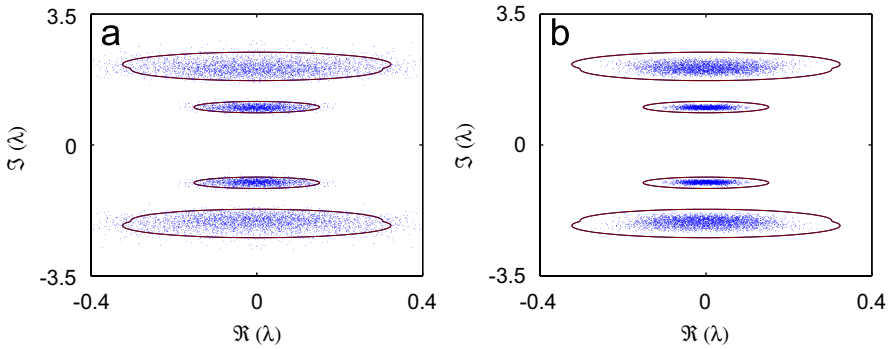

Fig. 4. Comparison of the structured pseudospectrum for $\varepsilon = 0.15$ and corresponding simulation results for normally distributed perturbations (a) and uniformly distributed perturbations (b) (see text for details).

k_i satisfy (20) but with the x_i being uniformly distributed over the complex unit circle. All the eigenvalues obtained from the simulations are now inside the pseudospectrum contour as expected. Observe also that the pseudospectrum contour is hardly approached. As pseudospectrum contours are related to a *worst-case* behaviour of the eigenvalues subjected to bounded perturbations, it seems unlikely to generate perturbations that push eigenvalues close to the boundary. Such an observation has also been made in Ref. [16].

3.2. An application from laser physics

In Ref. [5] pseudospectra have been applied to the analysis of the robust stability of a model for a semiconductor laser subject to optical feedback. For certain fixed model parameters, the problem leads to the study of the delay differential equation

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-1), \tag{23}$$

where

$$A_0 = \begin{bmatrix} -0.84982 & 0.14790 & 44.373 \\ 0.0037555 & -0.28049 & -229.23 \\ -0.17537 & 0.022958 & -0.36079 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0.28 & 0 & 0 \\ 0 & -0.28 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{24}$$

The stability of the zero solution of (23) is inferred from the eigenvalues, which are the zeros of the characteristic matrix

$$F(\lambda) = \lambda I - A_0 - A_1 \mathrm{e}^{-\lambda}. \tag{25}$$

As a characteristic of delay equations of retarded type, there are infinitely many eigenvalues, yet the number of eigenvalues in any right-half plane is finite [6]. Fig. 5 shows the rightmost eigenvalues of (23)–(24) computed with the software package DDE-BIFTOOL [3]. Notice the typical shape with a *tail* of eigenvalues to the left.

In this example we investigate the effect which an uncertainty on specific elements of A_0 and A_1 has on the eigenvalues by computing structured pseudospectra. From physical considerations an important requirement on the uncertainty is that in A_1 only the elements on positions (1,1) and (2,2) are non-zero and remain opposite to each other. Physically, these elements describe the feedback process of the laser; see Ref. [18] for full details. We can take this structure into account by considering perturbations on A_1 of the form $\mathrm{diag}(\delta a, -\delta a, 0)$, with $\delta a \in \mathbb{C}$, in addition to unstructured perturbations on A_0. The resulting additive uncertainty on F has the general form (5), namely

$$\delta F(\lambda) = \underbrace{-I_3}_{D_1(\lambda)} \delta A_0 \underbrace{I_3}_{E_1(\lambda)} + \delta a \underbrace{\begin{bmatrix} -1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}}_{G_1(\lambda)} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \mathrm{e}^{-\lambda}}_{H_1(\lambda)}. \tag{26}$$

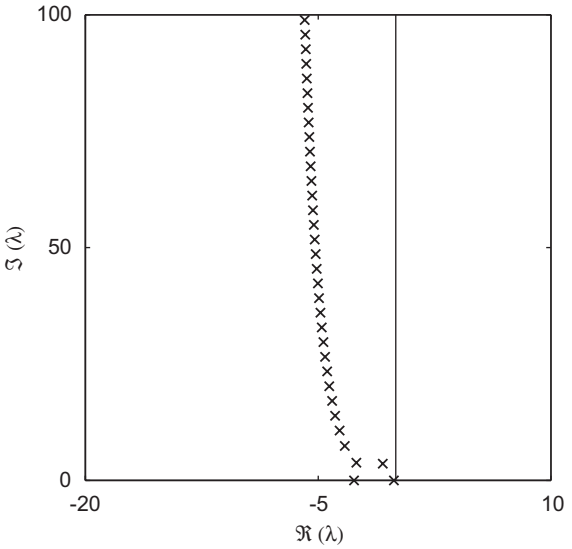

Fig. 5. Roots of (25) in the complex plane.

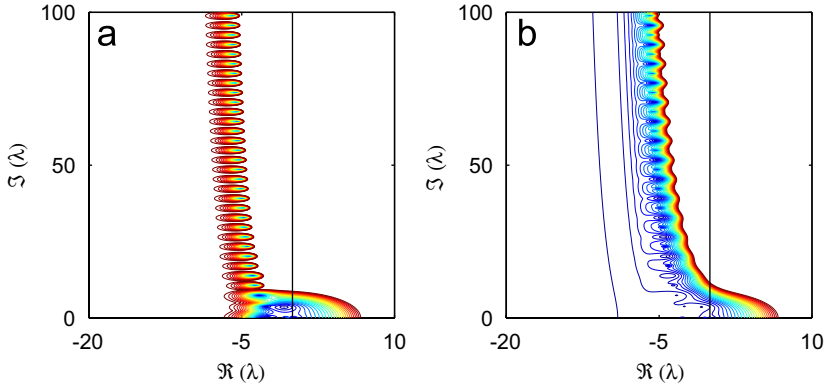

Fig. 6. Structured (a) and unstructured (b) pseudospectra of the delayed characteristic $F(\lambda)$ given by (25). From right to left (or from outermost to innermost), the contours correspond to $\varepsilon = 0.001$ to $\varepsilon = 0.27$ in intervals of 0.01.

An application of Theorem 1 yields

$$\Lambda_\varepsilon^{\mathrm{str}}(F) = \left\{ \lambda \in \mathbb{C} \,:\, \mu_{\mathbf{\Delta}} \left(\left[\begin{array}{c} I_3 \\ \hline \mathrm{e}^\lambda \;\; 0 \;\; 0 \\ 0 \;\; \mathrm{e}^\lambda \;\; 0 \end{array} \right] F(\lambda)^{-1} \left[\begin{array}{c|cc} & -1 & 0 \\ -I_3 & 0 & 1 \\ & 0 & 0 \end{array} \right] \right) > \frac{1}{\varepsilon} \right\},$$

where $\mathbf{\Delta}$ is the set of complex block-diagonal 5×5 matrices with one full 3×3 block and one repeated scalar 2×2 block. For this type of uncertainty structure ($f = s = 1$), the ssv can be computed exactly as the solution of a convex optimization problem; see the Appendix. We have once again combined the `mussv` routine of `MATLAB` with a contour plotter to visualize the structured pseudospectra and the results are shown in Fig. 6(a).

For comparison, unstructured pseudospectra of (25) in the sense of Ref. [5] are shown in Fig. 6(b). This corresponds to

$$\delta F(\lambda) = \delta A_0 + \delta A_1 \mathrm{e}^{-\lambda},$$

where δA_0 and δA_1 are unstructured. Thus, combining Theorems 1 and 2 one obtains

$$\Lambda_\varepsilon = \left\{ \lambda \in \mathbb{C} : \|F(\lambda)^{-1}\|_2 (1 + |\mathrm{e}^{-\lambda}|) > \frac{1}{\varepsilon} \right\}.$$

As a significant qualitative difference, the unstructured ε-pseudospectra stretch out infinitely far along the negative real axis, even for arbitrarily small values of ε. In Ref. [9, Section 3.3], this phenomenon is related to the behaviour of eigenvalues, which are introduced by perturbations that make the matrix A_1 non-singular. Such perturbations are, however, non-physical and, as we have shown, can be excluded by applying the novel structured uncertainty (26). Again, the computational cost of applying structured perturbations was found to be approximately 400 times that of the unstructured (non-physical) case [5]. However, it is worth mentioning the alternative way of computing pseudospectra in DDEs by considering a large system obtained from the discretization of the infinitesimal generator, which appears when the delay equation is reformulated as a first-order equation over an infinite-dimensional space [2]. The computational cost of our structured approach is only twice that of the (non-physical) pseudospectra computation for this large system [5].

4. Conclusions

We have presented a general theory for computing structured pseudospectra of nonlinear eigenvalue problems. Our novel method allows one to direct perturbations to specific elements (or, indeed, groups of elements) of the individual matrices of a corresponding eigenvalue problem.

As an illustration, we first applied these methods to an example from structural dynamics. In this case the eigenvalue problem was of second order. We showed how structured perturbations could be directly compared to probabilistic uncertainties on the parameters. The pseudospectra were used to derive bounds on the position of the eigenvalues obtained through a computationally intensive random simulation.

Our second example involved an infinite-dimensional eigenvalue problem obtained from the modelling of a feedback laser using delay differential equations. Here, structured perturbations were applied in order to preserve the structure of the matrix associated with the delayed variable. Specifically, in the governing system this matrix was singular. With the structured approach we could allow physically realistic perturbations only, which have the property of maintaining the singularity of the matrix. This leads to pseudospectra which are quantitatively and qualitatively different from the case where unstructured perturbations are allowed. This stems from the fact that the latter generically increase the rank of the matrix.

Acknowledgements

The research of Thomas Wagenknecht and Kirk Green was supported by the EPSRC grant GR/535684/01. This article presents results of the Project IUAP P5, funded by the Interuniversity Poles of Attraction-Belgian Federal Science Policy Office. Wim Michiels is a post-doctoral fellow of the Fund for Scientific Research - Flanders (Belgium).

Appendix A. The structured singular value

In this Appendix, we introduce the concept of ssv of matrices and outline the main principles behind the standard computational schemes. A more elaborate introduction can be found in the review paper [10], Ref. [20, Chapter 11] and Ref. [8, Chapter 4].

Let $G \in \mathbb{C}^{N\times M}$ and denote its highest singular value by $\sigma_1(G)$ (other singular values in descending order). A classical result from robust control theory, which lays the basis for the celebrated small gain theorem, relates the largest singular value of G to the solutions of the equation

$$\det(I + G\Delta) = 0 \tag{27}$$

in the following way:

$$\sigma_1(G) = \begin{cases} 0 & \text{if } \det(I + G\Delta) \neq 0 \ \forall \Delta \in \mathbb{C}^{M\times N}, \\ (\min\{\sigma_1(\Delta) : \Delta \in \mathbb{C}^{M\times N} \text{ and } \det(I + G\Delta) = 0\})^{-1} & \text{otherwise.} \end{cases} \tag{28}$$

We refer to Δ as the 'uncertainty'. As in a robust control framework, (27) typically originates from a feedback interconnection of a nominal transfer function and an uncertainty block.

Next we reconsider the solutions of Eq. (27), where Δ is restricted to having a particular structure by imposing $\Delta \in \mathbf{\Delta}$, with $\mathbf{\Delta}$ a closed subset of $\mathbb{C}^{M\times N}$. In analogy with (28) one defines the *ssv* of the matrix G with respect to the uncertainty set $\mathbf{\Delta}$ as

$$\mu_{\mathbf{\Delta}}(G) := \begin{cases} 0 & \text{if } \det(I + G\Delta) \neq 0 \ \forall \Delta \in \mathbf{\Delta}, \\ (\min\{\sigma_1(\Delta) : \Delta \in \mathbf{\Delta} \text{ and } \det(I + G\Delta) = 0\})^{-1} & \text{otherwise.} \end{cases} \tag{29}$$

It directly follows from the definition that

$$\mu_{\mathbf{\Delta}}(G) \leqslant \sigma_1(G). \tag{30}$$

Furthermore, if $\mathbb{C}\mathbf{\Delta} = \mathbf{\Delta}$, then

$$\mu_{\mathbf{\Delta}}(G) = \max_{\Delta\in\mathbf{\Delta},\ \sigma_1(\Delta)=1} \rho(G\Delta) \tag{31}$$

with $\rho(\cdot)$ the spectral radius.

In what follows we restrict ourselves, for simplicity, to an uncertainty set of form (9) with $\sum_{i=1}^{f} k_i + \sum_{i=1}^{s} m_i = M$ and $\sum_{i=1}^{f} l_i + \sum_{i=1}^{s} m_i = N$. Such a set satisfies $\mathbb{C}\mathbf{\Delta} = \mathbf{\Delta}$. Furthermore, based on a slight generalization of Ref. [10, Lemma 6.3] to non-square block-diagonal perturbations, the search space of the optimization in the right-hand side of (31) can be restricted. This results in

$$\mu_{\mathbf{\Delta}}(G) = \max_{U\in\mathscr{U}} \rho(GU), \tag{32}$$

where $\mathscr{U} \subseteq \mathbf{\Delta}$ is defined as

$$\begin{aligned} \mathscr{U} := \{&\operatorname{diag}(U_1, \ldots, U_f, u_1 I_{m_1}, \ldots, u_s I_{m_s}) : U_i \in \mathbb{C}^{k_i\times l_i},\ u_j \in \mathbb{C} \\ &\sigma_k(U_i) = 1,\ 1 \leqslant k \leqslant \min(k_i, l_i),\ |u_j| = 1,\ 1 \leqslant i \leqslant f,\ 1 \leqslant j \leqslant s\}. \end{aligned}$$

Note that the elements of $\mathscr{U}$ are *unitary* matrices if the uncertainty structure only involves square blocks, that is, $k_i = l_i$, $i = 1, \ldots, f$.

Next, the following invariance property can easily be checked:

$$\mu_{\mathbf{\Delta}}(G) = \mu_{\mathbf{\Delta}}(D_2 G D_1^{-1}) \quad \forall (D_1, D_2) \in \mathscr{D}, \tag{33}$$

where

$$\begin{aligned} \mathscr{D} := \{&(D_1, D_2) : D_1 = \operatorname{diag}(a_1 I_{k_1}, \ldots, a_f I_{k_f}, D_1, \ldots, D_s),\ D_2 \\ &= \operatorname{diag}(a_1 I_{l_1}, \ldots, a_f I_{l_f}, D_1, \ldots, D_s) : a_i > 0,\ D_i \in \mathbb{C}^{m_i\times m_i},\ D_i^* = D_i > 0\}. \end{aligned}$$

From (32) and the combination of (33) and (30) we finally obtain

$$\max_{U\in\mathscr{U}} \rho(GU) = \mu_{\mathbf{\Delta}}(G) \leqslant \min_{(D_1,D_2)\in\mathscr{D}} \sigma_1(D_2 G D_1^{-1}). \tag{34}$$

Therefore, *optimization* procedures are typically used to compute estimates for $\mu_{\mathbf{\Delta}}(G)$. The function $U \in \mathscr{U} \to \rho(GU)$ may have several local maxima and, for this, a local search for a maximum is not guaranteed to lead to $\mu_{\mathbf{\Delta}}(G)$ but to lower bounds. An appropriate formulation of the optimality condition enables algorithms which resemble power algorithms for computing eigenvalues and singular values; see Ref. [11] for an example. Although the convergence of such algorithms to $\mu_{\mathbf{\Delta}}(G)$ is not guaranteed either and they may converge to values corresponding to lower bounds on $\mu_{\mathbf{\Delta}}(G)$, they have proven their effectiveness in practice. The computation of the upper bound in (34) can be recast into a standard *convex optimization* problem. However, in general $\mu_{\mathbf{\Delta}}(G)$ is not equal to the upper bound. An exception to this holds if the number of blocks in the matrices belonging to the uncertainty set $\mathbf{\Delta}$ satisfies $f + 2s \leqslant 3$ and all the blocks are square, $k_i = l_i$, $i = 1, \ldots, f$.

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